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A NOTE ON A HYPERGEOMETRIC SERIES BASED SOLUTION TO THE BASEL PROBLEM

Arjun Kumar Rathie and Dongkyu Lim*

Abstract. In a very recent paper, Campbell [A Wilf-Zeilbeger based solution to the Basel problem with applications, Discrete Math. Lett. 10 (2022), 21-27], after re-writing the Basel series into a ${}_3F_2(1)$ hypergeometric series, pointed out that "it is not obvious as to have classically known hypergeometric identities for ${}_3F_2(1)$-series with free parameters such as those of Dixon's summation identity, Whipple's summation identity or Watson's summation identity could be used to determine a full proof for the closed form ${}_3F_2(1)$-Basel series". Thus the aim of this note is to provide the solution of the ${}_3F_2(1)$-Basel series via the above-mentioned three summation identities.

1. Introduction

It is well-known that the problem of determining a closed form evaluation for the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots,$$

which is known in the literature as the Basel problem, is one of the most famous problems in the history of mathematics. This problem was solved for the first time in 1735 by mathematician Euler and in terms of the Riemann zeta function, the famous formula $\zeta(2)=\frac{\pi^2}{6}$ is also due to Euler. Since then, from time to time, many elegant proofs of this famous formula have been discovered by mathematicians and researchers. For this, we refer [2, 4–41, 43, 44, 46, 47] and the references therein.

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*Corresponding author.
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On the other hand, the generalization of the well-known and useful Gauss's hypergeometric function ${}_pF_q$ is defined as [1, 3, 42, 45]

$$(1.1) \qquad {}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n\cdots(a_p)_n}{(b_1)_n\,(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!},$$

where $a_j(j=1,2,\cdots,p)$ and $b_j(j=1,2,\cdots,q)$ may be real or complex numbers with an exception that $b_j(j=1,2,\cdots,q)$ should not be zero or a negative integer, z being the variable of the series, and $(a)_n$ denotes the Pochhammer's symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number $a(\neq 0)$ by

$$(1.2) \qquad (a)_n = \begin{cases} a(a+1)\ldots(a+n-1), & n\in\mathbb{N}, \\ 1, & n=0. \end{cases}$$

Further, in terms of the well-known Gamma function, the Pochhammer's symbol $(a)_n$ is represented as

$$(1.3) \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The series (1.1) is convergent for all values of z for $|z|<\infty$ if $p\leq q$ and for all values of z for $|z|<1$ if $p=q+1$. Also, when $|z|=1$ with $p=q+1$, the series (1.1) is convergent absolutely if $\Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$.

For more details about this function, we refer standard texts of Rainville [42], Slater [45], Bailey [3] and Andrews [1].

It is interesting to mention that whenever a hypergeometric function ${}_2F_1$ or the generalized hypergeometric function ${}_pF_q$ reduces to the gamma function, the result is very useful from the point of view of applications. In literature, there exists a large number of summation theorems. However, in our present investigations, we shall mention the following three classical summation theorems.

Dixon summation theorem [45]

(1.4)

$$ {}_3F_2\left[\begin{array}{c} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{array};1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)},$$

provided $\Re(a-2b-2c) > -2$.

Whipple's summation theorem [45]
(1.5)

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ e,\ f \end{matrix}; 1\right] = \frac{2^{1-2c}\pi\Gamma(e)\Gamma(f)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}f\right)}, $$

provided $a+b=1$, $e+f=1+2c$, and $\Re(c)>0$.

Watson summation theorem [45]

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix}; 1\right] \\ &\quad = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \end{aligned} \tag{1.6} $$

provided $\Re(2c-a-b)>-1$.

In addition to this, we shall also use the following transformation formula for the series ${}_3F_2(1)$ due to Thomas [3]:
(1.7)

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} \alpha_1,\ \alpha_2,\ \alpha_3 \\ \beta_1,\ \beta_2 \end{matrix}; 1\right] \\ &\quad = \frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}{}_3F_2\left[\begin{matrix} \beta_1-\alpha_1,\ \beta_1-\alpha_2,\ \alpha_3 \\ \beta_1,\ \beta_1+\beta_2-\alpha_1-\alpha_2 \end{matrix}; 1\right], \end{aligned} $$

provided $\Re(\beta_2-\alpha_3)>0$ and $\Re(\beta_2+\beta_1-\alpha_1-\alpha_2-\alpha_3)>0$.

In a very recent paper, Campbell [7], after re-writing the Basel series

$$ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \tag{1.8} $$

into a ${}_3F_2(1)$ hypergeometric series in the form

$$ {}_3F_2\left[\begin{matrix} 1,\ 1,\ 1 \\ 2,\ 2 \end{matrix}; 1\right] = \frac{\pi^2}{6} \tag{1.9} $$

pointed out that "it is not obvious as to have classically known hypergeometric summation theorems for the series ${}_3F_2(1)$ with free parameters such as those of Dixon's summation identity (1.4), Whipple's summation identity (1.5) or Watson's summation identity (1.6) could be used to determine a full proof for the closed-form ${}_3F_2(1)$-Basel series (1.9)". Thus our aim of this note is to provide the solution of the ${}_3F_2(1)$-Basel series by employing the summation identities (1.4) to (1.6).

2. Derivation of Basel series

2.1. Via Dixon and Whipple identites

Denoting the left-hand side of (1.9) by S, we have

$$\begin{aligned} S &= {}_3F_2\left[\begin{matrix} 1,\ 1,\ 1 \\ 2,\ 2 \end{matrix};1\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}S \\ \Rightarrow \frac{3}{4}S &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ \Rightarrow S &= \frac{1}{3}\sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2} \\ &= \frac{1}{3}\sum_{n=0}^{\infty} \frac{\Gamma^2\left(n+\frac{1}{2}\right)}{\Gamma^2\left(n+\frac{3}{2}\right)} \end{aligned}$$

converting into Pochhammer symbols, we have

$$S = \frac{4}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n (1)_n}{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n n!} \quad (\text{since } (1)_n = n!).$$

Finally, using (1.1), we have

$$S = \frac{4}{3}\ {}_3F_2\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ 1 \\ \frac{3}{2},\ \frac{3}{2} \end{matrix};1\right]. \tag{2.1}$$

We now observe that the ${}_3F_2$ appearing on the right-hand side of (2.1) can be evaluated with the help of the Dixon's identity (1.4) by letting $a = 1$, $b = c = \frac{1}{2}$ and after little simplification, we easily arrive at the right-hand side of (1.9).

We also observe that the function ${}_3F_2$ appearing on the right-hand side of (2.1) can also be evaluated with the help of the Whipple's identity (1.5) by letting $a = b = \frac{1}{2}$, $c = 1$ and $e = f = \frac{3}{2}$ and, after some calculation, we easily arrive at the right-hand side of (1.9). This completes the proof of the Basel series via Dixon and Whipple identities.

2.2. Via Watson identity

In the result (2.1), if we apply the Thomas transformation (1.7) by taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, α_3=1 and $\beta_1 = \beta_2 = \frac{3}{2}$, then after little simplification, we have

$$S = \frac{2}{3} \;{}_3F_2\left[\begin{matrix} 1,\ 1,\ 1 \\ \frac{3}{2},\ 2 \end{matrix}\ ;1\right].$$

We now observe that the function ${}_3F_2$ can be evaluated with the help of Watson's identity by taking $a = b = c = 1$ and after little algebra, we easily arrive at the right-hand side of (1.9). This completes the proof of the Basel series via Watson identity.

Concluding remark :
In this note, we have provided the solution of the ${}_3F_2(1)$-Basel series via Dixon, Whipple, and Watson summation identities.

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Arjun Kumar Rathie
Department of Mathematics
Vedant College of Engineering & Technology,
(Rajasthan Technical University)
Bundi-323021, Rajasthan state, India
E-mail: arjunkumarrathie@gmail.com

Dongkyu Lim
Department of Mathematics Education
Andong National University
Andong 36729, Republic of Korea
E-mail: dklim@anu.ac.kr